Spacecraft Attitude Representations

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Summary

The direction cosine matrix or attitude matrix is the most fundamental representation of the attitude, but it is very inefficient:

- It has six redundant parameters
- it is difficult to enforce the six (orthogonality) constraints

The four-component quaternion representation is very convenient:

- It has only one redundant parameter
- it is easy to enforce the normalization constraint
- the attitude matrix is a homogeneous quadratic function of q
- quaternion kinematics are bilinear in q and ω.

Euler angles are extensively used:

- they often have a physical interpretation
- they provide a natural description of some spacecraft motions (COBE, MAP)
- but kinematics and attitude matrix involve trigonometric functions
- "gimbal lock" for certain values of the angles

Other minimum (three-parameter) representations:

- Gibbs vector is infinite for 180° rotations, but useful for analysis
- Modified Rodrigues Parameters are nonsingular, no trig functions
- Rotation vector $\boldsymbol{\phi}$ is nonsingular, but requires trig functions

Outline

This is intended to be a critical review of well-known material.

The best single reference is “A Survey of Attitude Representations,” by Malcolm Shuster, in *the Journal of the Astronautical Sciences*, Vol. 41, No. 4, October–December 1993, pp. 439–517.

We will cover the following representations:

Direction Cosine Matrix

Euler Axis/Angle

Euler Angles

Generalized Euler angles

Euler-Rodrigues Symmetric Parameters (Quaternion)

Gibbs Vector (Rodrigues Parameters, Cayley Parameterization)

Modified Rodrigues Parameters

Direction Cosine Matrix or Rotation Matrix

Say that an (abstract) vector $\mathbf{v}$ has a (column) representation $\mathbf{v}_r$ in a reference frame, and a (column) representation $\mathbf{v}_b$ in the spacecraft body frame.

These representations are related by $\mathbf{v}_b = R\mathbf{v}_r$, where R is a 3×3 matrix, known as the direction cosine matrix, the rotation matrix, or the attitude matrix A.

If we rotate through an intermediate frame, we have

$$\mathbf{v}_b = R_{bi}\mathbf{v}_i = R_{bi}(R_{ir}\mathbf{v}_r) = (R_{bi}R_{ir})\mathbf{v}_r, \quad \text{so} \quad R_{br} = R_{bi}R_{ir}.$$

Rotations don't commute, in general, $R_1R_2 \neq R_2R_1$.

The rotation matrix preserves lengths and angles, which means that it is orthogonal,

$R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$, where I is the 3×3 identity matrix.

This gives six constraints on R, since I is symmetric, so the 3×3 matrix R has six redundant components. This reflects the three-dimensionality of the rotation group, but this group has no globally nonsingular 1–1 three-dimensional parameterization.

Euler Axis/Angle

Consider the rotation of a vector $\mathbf{v}$ by angle ϕ about an axis $\mathbf{e}$.

The component $(\mathbf{e}\cdot\mathbf{v})\mathbf{e}$ of $\mathbf{v}$ along $\mathbf{e}$ remains unchanged.
The component $[\mathbf{v}-(\mathbf{e}\cdot\mathbf{v})\mathbf{e}]$ perpendicular to $\mathbf{e}$ is rotated.
A fraction $\cos\phi$ of the perpendicular component remains in the same direction.
A fraction $\sin\phi$ of it is rotated into the perpendicular direction, along $\mathbf{v}\times\mathbf{e}$.

Thus

$$R(\mathbf{e},\phi)\mathbf{v} = (\mathbf{e}\cdot\mathbf{v})\mathbf{e} + \cos\phi[\mathbf{v}-(\mathbf{e}\cdot\mathbf{v})\mathbf{e}] - \sin\phi(\mathbf{e}\times\mathbf{v}),$$ or

$$R(\mathbf{e},\phi) = (\cos\phi)I + (1-\cos\phi)\mathbf{e}\mathbf{e}^{\mathrm{T}} - \sin\phi[\mathbf{e}\times]$$

where $[\mathbf{e}\times] \equiv \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$ is the cross-product matrix.

We often combine the Euler axis and angle into a rotation vector $\boldsymbol{\phi} \equiv \mathbf{e}\phi$. All rotations can be mapped to points in a sphere of radius π in rotation vector space, where points at opposite ends of a diameter represent the same 180° rotation.

This can be written as a matrix exponential, $R(\mathbf{e},\phi) = \exp[\boldsymbol{\phi}\times]$.

Kinematics of the Direction Cosine Matrix

The rotation matrix at a time $t+\Delta t$ is related to the matrix at time t by

$$\begin{aligned} R(t+\Delta t) &= R(\mathbf{e},\omega\Delta t)R(t) = \{\cos(\omega\Delta t)I + [1-\cos(\omega\Delta t)]\mathbf{e}\mathbf{e}^{\mathrm{T}} - \sin(\omega\Delta t)[\mathbf{e}\times]\}R(t) \\ &= (I-\omega\Delta t[\mathbf{e}\times])R(t) + \text{order}(\Delta t^2) = (I-[\boldsymbol{\omega}\times]\Delta t)R(t) + \text{order}(\Delta t^2) \end{aligned}$$

Then by the fundamental definition of the derivative

$$\dot{R} = \lim_{\Delta t\to 0}\left\{\frac{R(t+\Delta t)-R(t)}{\Delta t}\right\} = \lim_{\Delta t\to 0}\left\{\frac{(I-[\boldsymbol{\omega}\times]\Delta t)R(t)-R(t)}{\Delta t}\right\} = -[\boldsymbol{\omega}\times]R(t).$$

This is a very convenient kinematic relation, bilinear in R and $\boldsymbol{\omega}$.

It's easy to see that this preserves the orthogonality relation

$$\mathrm{d}(R^{\mathrm{T}}R)/\mathrm{d}t = \dot{R}^{\mathrm{T}}R + R^{\mathrm{T}}\dot{R} = (-[\boldsymbol{\omega}\times]R)^{\mathrm{T}}R + R^{\mathrm{T}}(-[\boldsymbol{\omega}\times]R) = R^{\mathrm{T}}[\boldsymbol{\omega}\times]R - R^{\mathrm{T}}[\boldsymbol{\omega}\times]R = 0.$$

However, numerical errors can cause the direction cosine matrix to lose orthogonality, and it is not especially easy to restore it.

Euler Angle Parameterization

$$R(\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_3;\varphi,\vartheta,\psi) \equiv R(\hat{\mathbf{n}}_3,\psi)R(\hat{\mathbf{n}}_2,\vartheta)R(\hat{\mathbf{n}}_1,\varphi)$$

Conventional Euler rotations are designated by the three indices, for example:

$$R_{213}(\varphi,\vartheta,\psi) \equiv R(\hat{\mathbf{2}},\hat{\mathbf{1}},\hat{\mathbf{3}};\varphi,\vartheta,\psi) = R(\hat{\mathbf{3}},\psi)R(\hat{\mathbf{1}},\vartheta)R(\hat{\mathbf{2}},\varphi),$$

where the rotation axes are selected from the set of coordinate axes

$$\hat{\mathbf{1}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \hat{\mathbf{2}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{3}} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

To represent a general rotation, successive rotations cannot be about the same axis.

This leaves twelve sets of conventional Euler axes:

six symmetric sets: 121, 232, 313, 131, 212, and 323

six asymmetric sets: 123, 231, 312, 132, 213, and 321.

Generalized Euler Angles

This generalized choice of Euler axes was discovered by Paul Davenport ("Rotations About Nonorthogonal Axes," *AIAA Journal*, Vol. 11, No. 6, June 1973, pp. 853–857) and rediscovered by Malcolm Shuster and me.

In order to be able to represent a general rotation matrix A, it is necessary and sufficient that the parameterization be capable of mapping any unit vector $\hat{\mathbf{u}}$ into any other unit vector $\hat{\mathbf{v}}$.

That is, there must exist angles φ, ϑ, and ψ such that the equation

$$\hat{\mathbf{v}} = A\hat{\mathbf{u}} = R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \varphi, \vartheta, \psi)\hat{\mathbf{u}}$$

has a solution for given $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$, $\hat{\mathbf{u}}$, and $\hat{\mathbf{v}}$.

To show necessity, we can take $\hat{\mathbf{u}}$ equal to $\hat{\mathbf{n}}_1$ and only look at the component of $\hat{\mathbf{v}}$ along $\hat{\mathbf{n}}_3$:

$$\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_3^T R(\hat{\mathbf{n}}_3, \psi) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi)\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_3^T R(\hat{\mathbf{n}}_2, \vartheta)\hat{\mathbf{n}}_1 .$$

The last step follows from the invariance of the axis of rotation.

Necessary Condition (I)

Insert the explicit form of the rotation matrix

$$R(\hat{\mathbf{n}},\zeta) = \cos\zeta\, I_{3\times 3} - \sin\zeta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} + (1-\cos\zeta)\hat{\mathbf{n}}\hat{\mathbf{n}}^T$$

into $\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \hat{\mathbf{n}}_3^T R(\hat{\mathbf{n}}_2,\vartheta)\hat{\mathbf{n}}_1.$

This gives

$$\begin{aligned} \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} &= (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1) + \sin\vartheta\left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] + \cos\vartheta(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\cdot(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1) \\ &= \beta + B\cos(\vartheta - \lambda) \end{aligned}$$

with

$$\begin{aligned} \beta &\equiv (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1), \\ B &\equiv \left|\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3\right| \left|\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1\right|, \end{aligned}$$

and

$$\lambda \equiv \mathrm{ATAN2}\left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),\ (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\cdot(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right].$$

Necessary Condition (II)

The right side of $\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \beta + B\cos(\vartheta - \lambda)$ is always between $\beta - B$ and $\beta + B$.
Thus a solution will exist for ϑ only if $\beta - B \le \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} \le \beta + B$.
But $\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}}$ can assume any value between – 1 and + 1, so we must have

$$\beta \equiv (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1) = 0 \qquad \text{and} \qquad B \equiv |\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3|\,|\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1| = 1$$

This means that $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = 0$ and $\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 = 0$,

or equivalently that $\hat{\mathbf{n}}_2$ is perpendicular to both $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_3$.

Then

$$\lambda = \text{ATAN2}\left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),\ \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{n}}_1\right]$$

and

$$\hat{\mathbf{n}}_3 = \cos\lambda\, \hat{\mathbf{n}}_1 + \sin\lambda (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) = R(\hat{\mathbf{n}}_2, \lambda)\hat{\mathbf{n}}_1.$$

Thus λ is the angle of the rotation about $\hat{\mathbf{n}}_2$ that takes $\hat{\mathbf{n}}_1$ into $\hat{\mathbf{n}}_3$.

We also have $\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_1 = \beta + B\cos(\vartheta - \lambda) = \cos(\vartheta - \lambda)$.

Extraction Of Euler Angles From Rotation Matrix

First find ϑ from $\vartheta = \lambda + \sigma\,\mathrm{ACOS}\left(\hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_1\right)$, where $\sigma = \pm 1$.

Then find φ and ψ from either

$$\varphi = \mathrm{ATAN2}\left[\sigma\,\hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_2,\ -\sigma\,\hat{\mathbf{n}}_3^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right]$$

and

$$\psi = \mathrm{ATAN2}\Big[\cos\varphi (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_2 + \sin\varphi (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),$$
$$\cos\varphi\, \hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_2 + \sin\varphi\, \hat{\mathbf{n}}_2^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\Big],$$

or from

$$\psi = \mathrm{ATAN2}\left[\sigma\,\hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_1,\ -\sigma\,(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_1\right]$$

and

$$\varphi = \mathrm{ATAN2}\Big[\cos\psi\, \hat{\mathbf{n}}_2^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) + \sin\psi (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),$$
$$\cos\psi\, \hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_2 + \sin\psi\, (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_2\Big].$$

You can set an arbitrary value for $\mathrm{ATAN2}(0,0)$, if this appears in either case

Sufficiency Of This Parameterization (I)

The rotation matrix can be written as the product

$$\begin{aligned} R(\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_3;\varphi,\vartheta,\psi) &= R\big(R(\hat{\mathbf{n}}_2,\lambda)\hat{\mathbf{n}}_1,\psi\big)R(\hat{\mathbf{n}}_2,\vartheta)R(\hat{\mathbf{n}}_1,\varphi) \\ &= R(\hat{\mathbf{n}}_2,\lambda)R(\hat{\mathbf{n}}_1,\psi)R^T(\hat{\mathbf{n}}_2,\lambda)R(\hat{\mathbf{n}}_2,\vartheta)R(\hat{\mathbf{n}}_1,\varphi) \\ &= R(\hat{\mathbf{n}}_2,\lambda)R(\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_1;\varphi,\vartheta-\lambda,\psi). \end{aligned}$$

For this to parameterize a general rotation matrix A, there must be angles φ, $\vartheta' \equiv \vartheta - \lambda$, and ψ such that

$$R(\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_3;\varphi,\vartheta,\psi) = A,$$

or equivalently

$$R(\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2,\hat{\mathbf{n}}_1;\varphi,\vartheta',\psi) = R^T(\hat{\mathbf{n}}_2,\lambda)A.$$

It is sufficient to show that the matrix on the left side of this equation can take the vectors in some orthonormal basis into an arbitrary orthonormal triad. Take this basis to be $\{\hat{\mathbf{n}}_1,\ \hat{\mathbf{n}}_2,\ \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\}$.

Sufficiency Of This Parameterization (II)

Thus we must be able to find angles φ, ϑ', and ψ such that

$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi)\hat{\mathbf{n}}_1 = \hat{\mathbf{v}}_1$, where $\hat{\mathbf{v}}_1$ is an arbitrary unit vector, and

$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi)\hat{\mathbf{n}}_2 = \hat{\mathbf{v}}_2$, where $\hat{\mathbf{v}}_2$ is a unit vector perpendicular to $\hat{\mathbf{v}}_1$.

Then the proper orthogonality of R ensures that it maps $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ into $\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2$.

Now $\quad \hat{\mathbf{v}}_1 = \cos\vartheta'\,\hat{\mathbf{n}}_1 + \sin\vartheta'\sin\psi\,\hat{\mathbf{n}}_2 + \sin\vartheta'\cos\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)$

and $\quad \hat{\mathbf{v}}_2 = R(\hat{\mathbf{n}}_1, \psi)R(\hat{\mathbf{n}}_2, \vartheta')[\cos\varphi\,\hat{\mathbf{n}}_2 - \sin\varphi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)] = \cos\varphi\,\hat{\mathbf{u}}_1 + \sin\varphi\,\hat{\mathbf{u}}_2$,

where $\quad \hat{\mathbf{u}}_1 \equiv \cos\psi\,\hat{\mathbf{n}}_2 - \sin\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)$

and $\quad \hat{\mathbf{u}}_2 \equiv \sin\vartheta'\,\hat{\mathbf{n}}_1 - \cos\vartheta'\sin\psi\,\hat{\mathbf{n}}_2 - \cos\vartheta'\cos\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)$.

Since $\{\hat{\mathbf{n}}_1,\ \hat{\mathbf{n}}_2,\ \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\}$ is a basis and $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ form an orthogonal basis for the plane perpendicular to $\hat{\mathbf{v}}_1$, it is clear that φ, ϑ', and ψ can be chosen to take $\{\hat{\mathbf{n}}_1,\ \hat{\mathbf{n}}_2,\ \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\}$ into an arbitrary orthonormal triad.

Kinematics

The body-referenced angular velocity vector is given by

$$\boldsymbol{\omega} = \dot{\psi}\,\hat{\mathbf{n}}_3 + \dot{\vartheta} R(\hat{\mathbf{n}}_3,\psi)\hat{\mathbf{n}}_2 + \dot{\varphi} R(\hat{\mathbf{n}}_3,\psi) R(\hat{\mathbf{n}}_2,\vartheta)\hat{\mathbf{n}}_1 = R(\hat{\mathbf{n}}_3,\psi)[\hat{\mathbf{n}}' \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3]\begin{bmatrix}\dot{\varphi}\\ \dot{\vartheta}\\ \dot{\psi}\end{bmatrix},$$

with $\hat{\mathbf{n}}' \equiv R(\hat{\mathbf{n}}_2,\vartheta)\hat{\mathbf{n}}_1 = R(\hat{\mathbf{n}}_2,\vartheta-\lambda)\hat{\mathbf{n}}_3 = \cos(\vartheta-\lambda)\hat{\mathbf{n}}_3 - \sin(\vartheta-\lambda)(\hat{\mathbf{n}}_2\times\hat{\mathbf{n}}_3)$.

The inverse of this equation gives the derivatives of the Euler angles in terms of the angular velocity:

$$\begin{bmatrix}\dot{\varphi}\\ \dot{\vartheta}\\ \dot{\psi}\end{bmatrix} = [\hat{\mathbf{n}}' \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3]^{-1} R^T(\hat{\mathbf{n}}_3,\psi)\boldsymbol{\omega}, \qquad \text{where}$$

$$[\hat{\mathbf{n}}' \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3]^{-1} = [\hat{\mathbf{n}}'\cdot(\hat{\mathbf{n}}_2\times\hat{\mathbf{n}}_3)]^{-1}[\hat{\mathbf{n}}_2\times\hat{\mathbf{n}}_3 \vdots \hat{\mathbf{n}}_3\times\hat{\mathbf{n}}' \vdots \hat{\mathbf{n}}'\times\hat{\mathbf{n}}_2]^T$$
$$= [\sin(\vartheta-\lambda)]^{-1}[\hat{\mathbf{n}}_3\times\hat{\mathbf{n}}_2 \vdots \sin(\vartheta-\lambda)\hat{\mathbf{n}}_2 \vdots \sin(\vartheta-\lambda)\hat{\mathbf{n}}_3 - \cos(\vartheta-\lambda)(\hat{\mathbf{n}}_3\times\hat{\mathbf{n}}_2)]^T.$$

Relation To The Conventional Euler Angles

Conventional Euler angle sets are subset of generalized Euler angles:

Symmetric sets (121, 232, 313, 131, 212, and 323) have $\lambda = 0$,
Even permutation asymmetric sets (123, 231, and 312) have $\lambda = \pi/2$,
Odd permutation asymmetric sets (132, 213, and 321) have $\lambda = -\pi/2$.

The equations for the generalized Euler angles provide universal formulas for the conventional Euler angle sets, with these substitutions.

The generalized Euler angles have the same "gimbal lock" singularity as the conventional angle sets.

The kinematic equations for φ and ψ are singular when $\sin(\vartheta - \lambda) = 0$, reflecting the fact that the axes $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_3$ coincide.

These are the generalizations of the well-known singularities of the conventional Euler angles for $\cos\vartheta = 0$ or $\sin\vartheta = 0$.

Quaternion or Euler-Rodrigues Symmetric Parameters

$$q = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathbf{e}\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix}.$$

$$R(q) = (q_4^2 - |\mathbf{q}|^2)I + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q_4[\mathbf{q}\times]$$

Note that $|q|^2 = 1$, so the quaternion is a point on a sphere in 4-dimensional space. Any hemisphere contains all rotations, since q and $-q$ give the same rotation matrix, so this is a 2-1 representation of the rotation group. If we write the composition of two rotations as $R(q')R(q) = R(q' \otimes q)$, then the multiplication rule for quaternions is

$$q' \otimes q = \begin{bmatrix} \mathbf{q}' \\ q'_4 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} q_4\mathbf{q}' + q'_4\mathbf{q} - \mathbf{q}'\times\mathbf{q} \\ q'_4 q_4 - \mathbf{q}'\cdot\mathbf{q} \end{bmatrix}.$$

This is the opposite of the historical convention, which has the opposite sign of the cross product, so that $R(q')R(q) = R(q\,q')$. The advantage of the convention adopted here is that the order of quaternion multiplcation is the same as the order of mutiplication of direction cosine matrices.

Quaternion Kinematics

Letting $q' = \begin{bmatrix} \mathbf{e}\sin(\omega\Delta t/2) \\ \cos(\omega\Delta t/2) \end{bmatrix} = \begin{bmatrix} \mathbf{e}\omega\Delta t/2 \\ 1 \end{bmatrix} + \text{order}(\Delta t^2) = \begin{bmatrix} \boldsymbol{\omega}\Delta t/2 \\ 1 \end{bmatrix} + \text{order}(\Delta t^2)$,

we find that

$$\dot{q} = \lim_{\Delta t \to 0} \left\{ \frac{q' \otimes q - q}{\Delta t} \right\} = \lim_{\Delta t \to 0} \left\{ \frac{1}{\Delta t} \left(\begin{bmatrix} \boldsymbol{\omega}\Delta t/2 \\ 1 \end{bmatrix} \otimes q - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes q \right) \right\} = \frac{1}{2} \begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes q .$$

The great utility of quaternions arises from the fact that the product rule is bilinear in the two arguments, resulting in this particularly simple kinematics.

Of the other attitude representations, only the direction cosine matrix obeys such simple kinematics.

If quaternion normalization is lost in numerical computations (in kinematic propagation or a Kalman filter update, for example), it can be restored trivially by

$q = q/|q|$. This is much simpler than orthogonalizing an approximately

orthogonal direction cosine matrix.

Gibbs Vector or Rodrigues Parameters (1840)

$$\mathbf{g} \equiv \frac{\mathbf{q}}{q_4} = \frac{\mathbf{e}\sin(\phi/2)}{\cos(\phi/2)} = \mathbf{e}\tan(\phi/2), \quad \text{with the inverse relation} \quad q = \frac{1}{\sqrt{1+|\mathbf{g}|^2}}\begin{bmatrix}\mathbf{g}\\ 1\end{bmatrix}.$$

This is like a gnomonic projection of the sphere. Either hemisphere projects to the entire plane, and q and $-q$ project to the same point, so this is a 1-1 representation of the rotations. This parameterization has the multiplication rule.

$$\mathbf{g}' \otimes \mathbf{g} = (\mathbf{g}' + \mathbf{g} - \mathbf{g}' \times \mathbf{g})/(1 - \mathbf{g}' \cdot \mathbf{g}).$$

This is related to the Cayley parameterization (1843)

$$A = (I - [\mathbf{g}\times])(I + [\mathbf{g}\times])^{-1} = (I + [\mathbf{g}\times])^{-1}(I - [\mathbf{g}\times]),$$

which is easily seen to be orthogonal. The inverse of this is

$$[\mathbf{g}\times] = (I - A)(I + A)^{-1} = (I + A)^{-1}(I - A).$$

The Gibbs vector is infinite for 180° rotations (the equator of the 4-sphere).

Modified Rodrigues Parameters

$$\mathbf{p} \equiv \frac{\mathbf{q}}{1+q_4} = \frac{\mathbf{e}\sin(\phi/2)}{1+\cos(\phi/2)} = \mathbf{e}\tan(\phi/4), \quad \text{with the inverse} \quad q = \frac{1}{1+|\mathbf{p}|^2}\begin{bmatrix} 2\mathbf{p} \\ 1-|\mathbf{p}|^2 \end{bmatrix}.$$

This is like a stereographic projection of the sphere. One hemisphere projects to interior of the unit sphere in 3-dimensional **p** space, and the other hemisphere projects to the exterior of this sphere.

The two parameters $\mathbf{p}$ and $-\mathbf{p}/|\mathbf{p}|^2$ represent the same rotation. Thus the origin and the "point at infinity" both represent a zero rotation.

The Modified Rodrigues Parameters have the multiplication rule.

$$\mathbf{p}' \otimes \mathbf{p} = \left[\left(1-|\mathbf{p}|^2\right)\mathbf{p}' + \left(1-|\mathbf{p}'|^2\right)\mathbf{p} - 2\mathbf{p}' \times \mathbf{p}\right] / \left(1+|\mathbf{p}'|^2|\mathbf{p}|^2 - 2\mathbf{p}' \cdot \mathbf{p}\right).$$

These parameters were invented by Thomas F. Wiener (MIT dissertation, 1962) and rediscovered by Modi and Marandi (1987)